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1975 J. Phys. A: Math. Gen. 8 808

(http://iopscience.iop.org/0305-4470/8/5/016)

J. Phys. A: Math. Gen., Vol. 8, No. 5, 1975. Printed in Great Britain. © 1975

O(4, 2): an exact invariance algebra for the electron

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Received 15 October 1974

Abstract. It is shown that the Lie Algebra O(4, 2) is an exact invariance algebra of Dirac's equation. The elements of this algebra become equivalent to the generators of conformal transformations in the zero-mass or high energy limit, provided the helicity of the electron is held at an appropriate value. A picture is suggested of the electron as 'a massless spin $\frac{1}{2}$ particle in a space–time knot'.

1. Introduction

It is well known that while the wave equation for the neutrino is invariant under the conformal group of transformations, Dirac's equation for the electron is not. Therefore it is suprising to find an O(4, 2) invariance algebra for the latter. The representations of O(4, 2) involved are of the type usually associated with a massless particle with spin $\frac{1}{2}$, but the elements of this algebra are, of course, not the generators of conformal transformations. However, they are found to become equivalent to them in the zero-mass or high energy limit, provided these limits are taken in an appropriate way.

These results enable one to view in a new light the approach to conformal invariance of Dirac's equation in these limits, and they suggest a picture of the electron as 'a massless spin $\frac{1}{2}$ particle in a space–time knot'.

2. Theory

Consider four-component spinor functions $\psi(x)$, and define Dirac matrices γ_μ, $\mu = 0, 1, 2, 3$ such that

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu},$$

where the diagonal metric tensor $g_{\mu\nu}(= g^{\mu\nu})$ has $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. The generators of the conformal group are (Mack and Salam 1969)

$$\begin{aligned} P_\mu &= \mathrm{i}\partial/\partial x^\mu, \qquad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + \tfrac{1}{4}\mathrm{i}[\gamma_\mu, \gamma_\nu], \\ D &= x^\mu P_\mu + \tfrac{3}{2}\mathrm{i}, \\ K_\mu &= x_\mu(2D+\mathrm{i}) - x^\nu x_\nu P_\mu - \mathrm{i}\gamma^\nu x_\nu \gamma_\mu. \end{aligned} \tag{1}$$

They satisfy the commutation relations

$$
\begin{aligned}
&[D, J_{\mu\nu}] = [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \\
&[D, P_\mu] = -\mathrm{i}P_\mu, \qquad [D, K_\mu] = \mathrm{i}K_\mu \\
&[P_\mu, J_{\nu\rho}] = \mathrm{i}(g_{\mu\nu}P_\rho - g_{\mu\rho}P_\nu) \\
&[K_\mu, J_{\nu\rho}] = \mathrm{i}(g_{\mu\nu}K_\rho - g_{\mu\rho}K_\nu) \\
&[K_\mu, P_\nu] = -2\mathrm{i}(g_{\mu\nu}D + J_{\mu\nu})
\end{aligned}
\tag{2}
$$

as well as those of the Lorentz algebra whose elements are $J_{\mu\nu}$. *In toto* these commutation relations are characteristic of the Lie algebra O(4, 2).

Dirac's equation,

$$(\gamma^\mu P_\mu - m)\psi = 0, \tag{3}$$

is invariant under the Poincaré subgroup of transformations generated by P_μ and $J_{\mu\nu}$, but is not left invariant by dilatations generated by D, nor by special conformal transformations generated by K_μ. Only when $m = 0$ is invariance under all transformations in the conformal group secured. More precisely, the equations

$$
\begin{aligned}
\gamma^\mu P_\mu \psi &= 0 \\
\gamma_5 \psi &= \pm\psi
\end{aligned}
\tag{4}
$$

where $\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$, are conformal-invariant. Nevertheless it is possible to find, in the usual Hilbert space of solutions of equation (3), Hermitian operators $J_{\mu\nu}$, P'_μ, K'_μ and D' satisfying the O(4, 2) commutation relations. In other words O(4, 2) is an invariance algebra for Dirac's equation.

Taking the alternating tensor with $\epsilon_{0123} = -1$, define

$$
\begin{aligned}
J &= \tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma}, \\
R_\mu &= \tfrac{1}{2}\{P_\mu, J\} - \tfrac{1}{2}\{P^\nu, J_{\nu\mu}\}, \\
L_\mu &= \tfrac{1}{2}\{P_\mu, J\} + \tfrac{1}{2}\{P^\nu, J_{\nu\mu}\}.
\end{aligned}
\tag{5}
$$

Then the following relations hold. (Proofs of the less obvious ones are presented at the end of this note.)

$$
\begin{aligned}
&[J, J_{\mu\nu}] = [L_\mu, L_\nu] = [R_\mu, R_\nu] = 0, \\
&[J, L_\mu] = -\mathrm{i}L_\mu, \qquad [J, R_\mu] = \mathrm{i}R_\mu, \\
&[L_\mu, J_{\nu\rho}] = \mathrm{i}(g_{\mu\nu}L_\rho - g_{\mu\rho}L_\nu), \\
&[R_\mu, J_{\nu\rho}] = \mathrm{i}(g_{\mu\nu}R_\rho - g_{\mu\rho}R_\nu), \\
&[R_\mu, L_\nu] = -2\mathrm{i}(g_{\mu\nu}J + J_{\mu\nu})P^\sigma P_\sigma.
\end{aligned}
\tag{6}
$$

These commutation relations are very similar to those of O(4, 2), the important difference being the appearance of the factor $P^\sigma P_\sigma$ in the commutator of R_μ and L_ν. However, on the space of solutions to Dirac's equation this factor reduces to m^2. (Note that J, R_μ and L_μ commute with $P^\sigma P_\sigma$ and $\gamma^\sigma P_\sigma$.) This enables one to define a set of operators P'_μ, K'_μ and D' which, taken together with the $J_{\mu\nu}$, satisfy the O(4, 2) commutation relations. One may take $P'_\mu = m^{-1}L_\mu$, $K'_\mu = m^{-1}R_\mu$ and $D' = J$, but it is more interesting to consider the possibilities

$$(a) \qquad P'_\mu = L_\mu, \qquad K'_\mu = m^{-2}R_\mu, \qquad D' = J$$

and

$$(b) \qquad P'_\mu = R_\mu, \qquad K'_\mu = m^{-2}L_\mu, \qquad D' = -J. \tag{7}$$

It can be seen from the definitions (5) that these operators are Hermitian if P_μ and $J_{\mu\nu}$ are Hermitian. Notice that J is a pseudoscalar and that R_μ and L_μ are neither polar nor axial vectors, but $(R_\mu - L_\mu)$ is a polar vector and $(R_\mu + L_\mu)$ is an axial vector. The appearance of the pseudoscalar J (or $-J$) as the 'dilatation operator' D' is encouraging when one notes (Bracken 1973) that in the zero-mass case, for ψ satisfying equation (4),

$$D\psi = \pm J\psi. \tag{8}$$

As a result of parity-violating equations like (4) and (8), there is in the zero-mass case no way of distinguishing scalar from pseudoscalar, nor polar from axial vector.

What representations of O(4, 2) are given by the operators $J_{\mu\nu}$, P'_μ, K'_μ and D' in the usual Hilbert space of solutions of Dirac's equation? This question is easily answered. One notes that if V_μ is any four-vector operator in this space, then

$$[J, V_\mu] = -\mathrm{i}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}V^\sigma, \tag{9}$$

so that with the help of equation (6) one has

$$\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}L^\sigma = L_\mu, \qquad \epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}R^\sigma = -R_\mu.$$

Since $[L_\mu, L_\nu] = [R_\mu, R_\nu] = 0$, these equations imply that $L^\mu L_\mu = R^\mu R_\mu = 0$. Thus in the case (a) one has

$$P^{\mu\prime}P'_\mu = 0, \qquad \omega'_\mu = -\tfrac{1}{2}P'_\mu,$$

and in the case (b) one has

$$P^{\mu\prime}P'_\mu = 0, \qquad \omega'_\mu = +\tfrac{1}{2}P'_\mu,$$

where $\omega'_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^{\sigma\prime}$ is the Pauli–Lubanski vector in each case. In other words the representation of O(4, 2) is, for either choice (a) or (b), of the type usually associated with a massless particle with spin $\frac{1}{2}$. In the case (a) this 'particle' has helicity $-\frac{1}{2}$, while in the case (b) it has helicity $+\frac{1}{2}$. Other relations characteristic (Barut and Böhm 1970) of these particular representations of O(4, 2) can be shown to hold. In particular,

$$K'_\mu P'_\nu + K'_\nu P'_\mu = J_{\mu\rho}J^\rho{}_\nu + J_{\nu\rho}J^\rho{}_\mu - 2\mathrm{i}(D' - \tfrac{3}{4}\mathrm{i})g_{\mu\nu} \tag{10}$$

in either case (a) or (b).

There are in fact two irreducible representations of O(4, 2) involved in each case. The operators P'_μ, K'_μ and D', constructed from the P_ν and $J_{\nu\rho}$, do not mix the subspaces corresponding to positive and negative eigenvalues of P_0, so that there is a representation in each subspace. It can be seen from what follows that P'_0 is positive-definite on positive energy solutions of Dirac's equation and negative-definite on negative energy solutions.

Now consider the zero-mass and high energy limits—that is to say the behaviour of P'_μ, K'_μ and D' on normalized functions ψ, such that

$$m^{-2}\boldsymbol{P}\,.\,\boldsymbol{P}\psi \to \infty, \tag{11}$$

where $\boldsymbol{P} = (P^1, P^2, P^3)$. In the case (a) one finds that the operators P'_μ and D' remain well defined in the limit, but that K'_μ remains well defined only if the functions ψ correspond to states of the electron with helicity $-\frac{1}{2}$. In other words one must impose the equation

$$\boldsymbol{\sigma}\,.\,\boldsymbol{P}\psi = -|\boldsymbol{P}|\psi,$$

where $\boldsymbol{\sigma} = (\gamma_2\gamma_3, \gamma_3\gamma_1, \gamma_1\gamma_2)$, while the limit is considered. Equivalently one requires that, in addition to (11),

$$\gamma_5\psi \to \psi. \tag{12}$$

Subject to this constraint, the operators P'_μ, K'_μ and D' tend towards operators equivalent to P_μ, K_μ and D. In particular it is clear from what has been said above that $D'\psi \to D\psi$.

Similarly if the O(4, 2) operators (*b*) are chosen, one can consider the limit

$$m^{-2}\boldsymbol{P}\,.\,\boldsymbol{P}\psi \to \infty, \qquad \gamma_5\psi \to -\psi, \tag{13}$$

and again P'_μ, K'_μ and D' tend towards operators equivalent to P_μ, K_μ and D. In particular it will again be true that $D'\psi \to D\psi$.

In more detail, in the case (*a*) one has from equation (21),

$$P'_\mu = -G_\mu + P_\mu J + \tfrac{1}{2}\mathrm{i}P_\mu(1-\gamma_5) - \tfrac{1}{2}\mathrm{i}(1-\gamma_5)\gamma_\mu\gamma^\nu P_\nu,$$

where

$$G_\mu = x_\mu P^\nu P_\nu - P_\mu(D-\mathrm{i}),$$

so that for ψ satisfying the conditions (11) and (12) one has

$$P'_\mu\psi \to P_\mu(2D-\mathrm{i})\psi = (2D+\mathrm{i})P_\mu\psi. \tag{14}$$

Then from the identity (10) one has

$$K'_0P'_0 = \boldsymbol{N}^2 - \mathrm{i}D' - \tfrac{3}{4}$$

where $\boldsymbol{N} = (J_{01}, J_{02}, J_{03})$, so that

$$K'_0 = (\boldsymbol{N}^2 - \mathrm{i}D' - \tfrac{3}{4})(P'_0)^{-1},$$

implying that

$$\begin{aligned} K'_0\psi &\to (\boldsymbol{N}^2 - \mathrm{i}D - \tfrac{3}{4})(2D-\mathrm{i})^{-1}(P_0)^{-1}\psi \\ &= (2D-\mathrm{i})^{-1}(\boldsymbol{N}^2 - \mathrm{i}D - \tfrac{3}{4})(P_0)^{-1}\psi. \end{aligned}$$

However, on the limiting functions, which are wavefunctions for a massless particle with helicity $-\frac{1}{2}$, one has

$$K_0P_0 = (\boldsymbol{N}^2 - \mathrm{i}D - \tfrac{3}{4}),$$

since the conformal group generators form a similar representation of O(4, 2) on such functions. Hence we conclude that

$$K'_0\psi \to (2D-\mathrm{i})^{-1}K_0\psi,$$

and by Lorentz covariance that

$$K'_\mu\psi \to (2D-\mathrm{i})^{-1}K_\mu\psi = K_\mu(2D+\mathrm{i})^{-1}\psi. \tag{15}$$

Now the transformation (in the Hilbert space of wavefunctions for a massless particle with helicity $-\frac{1}{2}$), which carries P_μ into $(2D+\mathrm{i})P_\mu$ and K_μ into $(2D-\mathrm{i})^{-1}K_\mu$, is a unitary transformation $U(D)$ defined by

$$U(D) = (2D+\mathrm{i})U(D+\mathrm{i}), \qquad U(D)U(D)^\dagger = 1. \tag{16}$$

For then

$$U(D)P_\mu U(D)^\dagger = P_\mu U(D-\mathrm{i})U(D)^\dagger = P_\mu(2D-\mathrm{i})$$

and

$$U(D)K_\mu U(D)^\dagger = K_\mu U(D+\mathrm{i})U(D)^\dagger = K_\mu(2D+\mathrm{i})^{-1}. \tag{17}$$

It is evident that $UDU^\dagger = D$ and that $UJ_{\mu\nu}U^\dagger = J_{\mu\nu}$, so that one may conclude that the O(4, 2) operators P'_μ, K'_μ, D' and $J_{\mu\nu}$ become equal to $UP_\mu U^\dagger$, $UK_\mu U^\dagger$, $UDU^\dagger$ and $UJ_{\mu\nu}U^\dagger$ in the limit on negative helicity wavefunctions. (Alternatively one could consider instead of P'_μ, K'_μ, D' and $J_{\mu\nu}$, the operators $U(D')^\dagger P'_\mu U(D')$, $U(D')^\dagger K'_\mu U(D')$, D' and $J_{\mu\nu}$, which also form an O(4, 2) invariance algebra for Dirac's equation and which become equal to the conformal group generators in this limit.) The treatment of the limit (13), with the operators as in (*b*), is entirely analogous. Note that in each case P'_0 is in the limit related to P_0 by a unitary transformation, so that the signs of P'_0 and P_0 are equal in the limit. Since both signs are invariant under the action of the O(4, 2) algebra they must be equal on all solutions of Dirac's equation.

3. Conclusion

These results enable one to view in a new way the approach to conformal-invariance of Dirac's equation in the zero-mass or high energy limit. On the one hand we have the widely discussed view (see for example Kastrup 1966) that the conformal group is an approximate invariance group for the electron at large values of $m^{-2}\boldsymbol{P}.\boldsymbol{P}$, and that D and K_μ are approximately constants of the motion in that situation. Now a second picture appears. There is an exact O(4, 2) invariance for the electron, but the energy–momentum four-vector P_μ is not in this Lie algebra. At large values of $m^{-2}\boldsymbol{P}.\boldsymbol{P}$, provided the helicity of the electron is kept fixed and the O(4, 2) algebra is chosen to match the helicity, this O(4, 2) is approximately equivalent to the Lie algebra of the conformal group (containing P_μ).

In either case, (*a*) or (*b*), the representation of O(4, 2) by P'_μ, K'_μ, D' and $J_{\mu\nu}$ is irreducible (for a given sign of P'_0) and occupies the same Hilbert space as the unitary representation of the Poincaré group, generated by P_μ and $J_{\mu\nu}$ (with the same sign for P_0). Therefore it must be possible not only to express P'_μ, K'_μ and D' in terms of the P_ν and $J_{\nu\rho}$, as done above, but also conversely to express P_μ in terms of the P'_ν, K'_ν, D' and $J_{\nu\rho}$. Such an expression clearly would involve m. Suppose one now regards P'_μ, K'_μ, D' and $J_{\mu\nu}$ as the generators of 'conformal transformations' for a massless spin $\frac{1}{2}$ 'particle' (with helicity $-\frac{1}{2}$ in case (*a*), $+\frac{1}{2}$ in case (*b*)) moving in a peculiar space, with coordinates different from x_μ, the coordinates of Minkowski space. (The structure of the O(4, 2) operators as given above indicates that the coordinates in this peculiar space transform as a four-vector under Lorentz transformations of Minkowski space, but transform as neither a polar nor an axial vector under a parity transformation of Minkowski space.) Given that there is an expression for the energy–momentum four-vector of the electron as a function of these 'conformal group' generators, one may then think of the electron as a massless spin $\frac{1}{2}$ particle in a sort of 'space–time knot'. The complication or extent of this knot is characterized by m or by the Compton wavelength of the electron. The underlying massless particle can be taken to have either helicity, $-\frac{1}{2}$ or $+\frac{1}{2}$, and one has a picture of the knot unravelling as the value of $m^{-2}\boldsymbol{P}\cdot\boldsymbol{P}$ increases, provided the helicity of the electron is kept fixed, and the helicity of the underlying particle is chosen to have that same value.

Appendix

In order to prove that $[J, L_\mu] = -\mathrm{i}L_\mu, [J, R_\mu] = \mathrm{i}R_\mu$, it is necessary and sufficient to show that

$$[J, \{P_\mu, J\}] = -\mathrm{i}\{P^\nu, J_{\nu\mu}\} \tag{18}$$

and that

$$[J, \{P^\nu, J_{\nu\mu}\}] = -\mathrm{i}\{P_\mu, J\}. \tag{19}$$

In the former case one notes that

$$\{P^\nu, J_{\nu\mu}\} = -\mathrm{i}[\tfrac{1}{2}J_{\nu\rho}J^{\nu\rho}, P_\mu]$$

and that

$$[J, \{P_\mu, J\}] \equiv [J^2, P_\mu].$$

Then equation (18) follows because, as may readily be checked,

$$J^2 = -J_{\mu\nu}J^{\mu\nu} - \tfrac{3}{4}. \tag{20}$$

Now note that

$$\{P^\nu, J_{\nu\mu}\} = 2P^\nu J_{\nu\mu} - 3\mathrm{i}P_\mu$$

so that, using equation (9),

$$[J, \{P^\nu, J_{\nu\mu}\}] = 2\mathrm{i}P_\tau(J^{\tau\sigma}\epsilon_{\sigma\mu\nu\rho}J^{\nu\rho}) - 3\epsilon_{\mu\nu\rho\sigma}P^\sigma J^{\nu\rho}.$$

But it follows from the commutation relations for the Lorentz group,

$$\mathrm{i}[J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho} - g_{\nu\rho}J_{\mu\sigma} - g_{\mu\sigma}J_{\nu\rho},$$

that

$$J^{\tau\sigma}\epsilon_{\sigma\mu\nu\rho}J^{\nu\rho} = \mathrm{i}g^{\tau\sigma}\epsilon_{\sigma\mu\nu\rho}J^{\nu\rho} - J\delta^\tau{}_\mu,$$

so that

$$[J, \{P^\nu, J_{\nu\mu}\}] = -\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma - 2\mathrm{i}P_\mu J = -\mathrm{i}\{P_\mu, J\},$$

and equation (19) is proved.

Next note that, from the definitions,

$$\begin{aligned} L_\mu &= P_\mu J - \tfrac{1}{2}\mathrm{i}\epsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma - J_{\mu\nu}P^\nu + \tfrac{3}{2}\mathrm{i}P_\mu \\ &= P_\mu J + \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho P^\sigma - x_\mu P^\nu P_\nu + x^\nu P_\nu P_\mu + \tfrac{1}{4}\mathrm{i}P^\nu[\gamma_\nu, \gamma_\mu] + \tfrac{3}{2}\mathrm{i}P_\mu. \end{aligned}$$

Now

$$\epsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho = \mathrm{i}\gamma_5[\gamma_\mu, \gamma_\sigma] = 2\mathrm{i}\gamma_5(\gamma_\mu\gamma_\sigma - g_{\mu\sigma}),$$

so that

$$L_\mu = P_\mu J + \mathrm{i}P_\mu\zeta_- - \mathrm{i}\gamma_\mu\gamma^\nu P_\nu\zeta_- - G_\mu \tag{21}$$

where

$$\zeta_{\pm} = \tfrac{1}{2}(1 \pm \gamma_5)$$

and

$$G_\mu = x_\mu P^\nu P_\nu - x^\nu P_\nu P_\mu - \tfrac{3}{2}\mathrm{i}P_\mu = x_\mu P^\nu P_\nu - DP_\mu. \tag{22}$$

Similarly,

$$R_\mu = P_\mu J - \mathrm{i}P_\mu\zeta_+ + \mathrm{i}\gamma_\mu\gamma^\nu P_\nu\zeta_+ + G_\mu.$$

Then, noting that $\zeta_+\zeta_- = \zeta_-\zeta_+ = 0$, one has

$$\begin{aligned} R_\mu L_\nu &= P_\mu L_\nu(J-\mathrm{i}) - \mathrm{i}P_\mu\zeta_+ L_\nu + \mathrm{i}\gamma_\mu\gamma^\nu P_\nu\zeta_+ L_\nu + G_\mu L_\nu \\ &= P_\mu P_\nu(J^2 - 2\mathrm{i}J\zeta_+ + \zeta_-) - P_\mu G_\nu(J - \mathrm{i} - \mathrm{i}\zeta_+) + G_\mu P_\nu(J + \mathrm{i}\zeta_-) \\ &\quad - \mathrm{i}P_\mu\gamma_\nu\gamma^\sigma P_\sigma(J-\mathrm{i})\zeta_- + \mathrm{i}\gamma_\mu P_\nu\gamma^\sigma P_\sigma J\zeta_+ - \mathrm{i}\gamma_\mu\gamma^\sigma P_\sigma G_\nu\zeta_+ - \mathrm{i}G_\mu\gamma_\nu\gamma^\sigma P_\sigma\zeta_- - G_\mu G_\nu \end{aligned} \tag{23}$$

and

$$\begin{aligned} L_\nu R_\mu = {} & P_\nu P_\mu(J^2 + 2\mathrm{i}J\zeta_- + \zeta_+) + P_\nu G_\mu(J + \mathrm{i} + \mathrm{i}\zeta_-) - G_\nu P_\mu(J - \mathrm{i}\zeta_+) + \mathrm{i}P_\nu\gamma_\mu\gamma^\sigma P_\sigma(J+\mathrm{i})\zeta_+ \\ & - \mathrm{i}\gamma_\nu P_\mu\gamma^\sigma P_\sigma J\zeta_- - \mathrm{i}\gamma_\nu\gamma^\sigma P_\sigma G_\mu\zeta_- - \mathrm{i}G_\nu\gamma_\mu\gamma^\sigma P_\sigma\zeta_+ - G_\nu G_\mu. \end{aligned}$$

From this point it is a straightforward matter to evaluate $[R_\mu, L_\nu]$, provided one notes that

$$\begin{aligned} &[G_\mu, G_\nu] = \mathrm{i}L_{\mu\nu}P^\sigma P_\sigma \\ &G_\mu P_\nu - G_\nu P_\mu = L_{\mu\nu}P^\sigma P_\sigma \\ &[G_\mu, P_\nu] = \mathrm{i}(P_\mu P_\nu - g_{\mu\nu}P^\sigma P_\sigma), \end{aligned} \tag{24}$$

where

$$L_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu.$$

In order to show that $[L_\mu, L_\nu] = 0$ one notes that

$$\begin{aligned} L_\mu L_\nu = {} & P_\mu P_\nu(J^2 + 2\mathrm{i}J\zeta_- - \mathrm{i}J) - P_\mu G_\nu(J + \mathrm{i}\zeta_- - \mathrm{i}) - G_\mu P_\nu(J + \mathrm{i}\zeta_-) - \mathrm{i}P_\mu\gamma_\nu\gamma^\sigma P_\sigma J\zeta_- \\ & - \mathrm{i}\gamma_\mu P_\nu\gamma^\sigma P_\sigma(J+\mathrm{i})\zeta_- - \gamma_\mu\gamma^\sigma P_\sigma\gamma_\nu\gamma^\sigma P_\sigma\zeta_- + \mathrm{i}\gamma_\mu\gamma^\sigma P_\sigma G_\nu\zeta_- + \mathrm{i}\gamma_\nu G_\mu\gamma^\sigma P_\sigma\zeta_- + G_\mu G_\nu, \end{aligned}$$

and again the proof is straightforward from this point, with the help of equations (24). The proof that $[R_\mu, R_\nu] = 0$ is quite similar.

The remaining relations in the set (6) are simply statements that J is a Lorentz (pseudo-) scalar and that L_μ and R_μ are four-vector operators, and as such are obviously correct.

In order to prove the identity (10) it is sufficient to show that

$$R_{(\mu}L_{\nu)} = (J_{\mu\alpha}J^\alpha{}_\nu + J_{\nu\alpha}J^\alpha{}_\mu)P^\sigma P_\sigma - 2\mathrm{i}g_{\mu\nu}(J - \tfrac{3}{4}\mathrm{i})P^\sigma P_\sigma, \tag{25}$$

where $R_{(\mu}L_{\nu)}$ denotes $R_\mu L_\nu + R_\nu L_\mu$. Now from equation (23),

$$\begin{aligned} R_{(\mu}L_{\nu)} = {} & -G_{(\mu}G_{\nu)} + 2\mathrm{i}G_{(\mu}P_{\nu)} + 2P_\mu P_\nu(J^2 - \mathrm{i}\gamma_5 J + 2) - 2\mathrm{i}g_{\mu\nu}(J-\mathrm{i})P^\sigma P_\sigma \\ & + \mathrm{i}\gamma_{(\mu}P_{\nu)}\gamma^\sigma P_\sigma(\gamma_5 J + \mathrm{i}) - \mathrm{i}\gamma_{(\mu}G_{\nu)}\gamma^\sigma P_\sigma. \end{aligned} \tag{26}$$

It is not difficult to verify that

$$J^2 - i\gamma_5 J = -\tfrac{1}{2}L_{\mu\nu}L^{\mu\nu} - \tfrac{3}{4}, \tag{27}$$

that

$$\gamma_{(\mu}G_{\nu)}\gamma^\sigma P_\sigma - \gamma_{(\mu}P_{\nu)}\gamma^\sigma P_\sigma(\gamma_5 J + i) = -(L_{\mu\sigma}\gamma^\sigma\gamma_\nu + L_{\nu\sigma}\gamma^\sigma\gamma_\mu + 2ig_{\mu\nu})P^\rho P_\rho \tag{28}$$

and that

$$-G_{(\mu}G_{\nu)} + 2iG_{(\mu}P_{\nu)} - 2P_\mu P_\nu(\tfrac{1}{2}L_{\rho\sigma}L^{\rho\sigma} - \tfrac{5}{4}) = (L_{\mu\sigma}L^\sigma{}_\nu + L_{\nu\sigma}L^\sigma{}_\mu + g_{\mu\nu})P^\sigma P_\sigma. \tag{29}$$

On combining equations (26)–(29), one obtains equation (25) as required.

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